NONSTEADY-STATE PROPAGATION OF A SYSTEM OF CRACKS IN A BRITTLE MATERIAL

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Abstract: An approximate equation for the nonsteady-state propagation of rectilinear cracks was derived and the problem of the equilibrium and propagation of a symmetrical systems of cracks in an elastic-brittle material was analyzed.

1. **Approximate equation of crack propagation.** There have been very few published papers [1–3] in which crack propagation is analyzed with the aid of exact solutions of the dynamic equations of the theory of elasticity. Due to considerable mathematical difficulties, the authors of these papers usually started from such artificial formulations of the problem that the solutions obtained do not lend themselves to physical interpretation. It is therefore natural that many attempts have been made to obtain an approximate description of crack propagation in terms more closely corresponding to real conditions. One of the first studies of this kind was carried out by Mott [4] who, having supplemented the usual energy equation of equilibrium for a crack [5]

$$\partial W / \partial l = \partial \Pi / \partial l \tag{1.1}$$

with a derivative $\partial T/\partial l$ (where T is kinetic energy and l the crack length),

$$\partial W / \partial l = \partial \Pi / \partial l + \partial T / \partial l, \tag{1.2}$$

obtained a simple formula

$$V = k\sqrt{E/\rho}\,(1 - l_0/l)^{1/2}. \tag{1.3}$$

Here W is the potential energy of deformation, Π is the surface energy, E is Young's modulus, ρ is the density, l_0 is the length of an equilibrium crack, and k is an empirical constant. Mott used a static expression for the potential energy of deformation which appears to be justified, since experiments carried out at a much later date by Wells and Post [6] showed that the stressed state in the vicinity of a growing crack is not substantially different from the static case; the same investigation showed that formula (1.3) is in good qualitative agreement with experimental data though it slightly overestimates the velocity (at k = = 0.38). It should be noted that Mott's analysis is based on an approximate formulation of the problem and that Eq. (1.2) does not satisfy the conservation energy law.

In fact, if the work done by external forces is denoted by A, we should have

$$\frac{dA}{dt} = \frac{dW}{dt} + \frac{dT}{dt} + \frac{d\Pi}{dt}. \tag{1.4}$$

Since the zone boundary varies with time, $d/dt = \partial/\partial t + \partial/\partial l\, V$, where $\partial/\partial t$ denotes a derivative with respect to time at a constant boundary. It is evident that

$$\frac{\partial A}{\partial t} = \frac{\partial W}{\partial t} + \frac{\partial T}{\partial t}, \quad \frac{d\Pi}{dt} = \frac{\partial \Pi}{\partial l} V. \tag{1.5}$$

On the other hand, in accordance with the theorem of the potential energy of deformation [7] in the plane case,

$$A = 2W + \iint \rho \left(u \frac{\partial^2 u}{\partial t^2} + v \frac{\partial^2 v}{\partial t^2}\right) dx\, dy, \tag{1.6}$$

where u and v are components of the displacement vector. Combining (1.4)–(1.6), we obtain

$$\frac{\partial W}{\partial l} = \frac{\partial T}{\partial l} + \frac{\partial \Pi}{\partial l} - \frac{\partial}{\partial l}\iint \rho \left(u \frac{\partial^2 u}{\partial t^2} + v \frac{\partial^2 v}{\partial t^2}\right) dx\, dy. \tag{1.7}$$

It will be seen that this expression differs from (1.2) in that it has an additional term on the right-hand side.

Let us consider the following problem. Let there be in an elastic material, an equilibrium crack whose length and tensile load at infinity satisfy the following condition:

$$N = K. \tag{1.8}$$

Here N is the stress intensity coefficient and K denotes the constriction cohesion coefficient.

At the initial instant $t = t_0$, the tensile force instantaneously increases to a level $p > p_0$. The problem is to determine the crack propagation velocity. First of all, we reject the condition of the smooth joining of the crack edges and retain only the physically necessary condition of the finiteness of stress.

The following assumptions are made. 1) The displacement vector components at any moment are determined in the same way as in the static problem; 2) the different between the applied external forces and the cohesive forces in the crack tip is balanced by forces of inertia.

Let us consider a particle of the material adjacent to the internal crack surface at a point a small distance s from the crack tip. If the velcoity of the progressive movement of the tip region is V, (accurate to small values of higher order) at this point we have

$$\frac{\partial v}{\partial t} = V \frac{\partial v}{\partial s}. \tag{1.9}$$

This velocity is attained by the particle in a time on the order of s/V, so that the acceleration

$$\frac{\partial^2 v}{\partial t^2} \sim \frac{V^2}{s}\frac{\partial v}{\partial s} \qquad \left(V = 4\frac{1-\nu^2}{E} N s^{1/2}\right). \tag{1.10}$$

In accordance with the first assumption, we obtain

$$\frac{\partial^2 v}{\partial t^2} = \frac{2V^2(1-\nu^2)}{E}\frac{N}{s^{3/2}}. \tag{1.11}$$

It is easy to show that the derivative $\partial^2 v/\partial x^2$ has a value of the same order of magnitude. In fact, according to [8]

$$v = \frac{4(1-\nu^2)}{E}\operatorname{Im}\varphi(z), \qquad \varphi(z) = \frac{p}{2}\sqrt{z^2 - l^2}. \tag{1.12}$$

Consequently, at $|x| \le l$,

$$\frac{\partial^2 v}{\partial x^2} = \frac{2p(1-\nu^2)}{E(l^2 - x^2)^{1/2}}\left[\frac{x^2}{l^2 - x^2} - 1\right]. \tag{1.13}$$

At the distance $s = l - x \ll l$ from the crack tip, $\partial^2 v/\partial x^2$ has a value on the order of

$$N(1-\nu^2)/Es^{3/2}.$$

In purely formal terms, the dynamic equation,

$$(\lambda + \mu)\frac{\partial \theta}{\partial y} + \mu \Delta v = \rho \frac{\partial^2 v}{\partial t^2}, \tag{1.14}$$

becomes static in the vicinity of the crack tip if N is replaced by

$$N\left[1 - k\frac{V^2 \rho}{\mu}\right], \tag{1.15}$$

where k is an empirical factor.

Using the second assumption, we take

$$N - K = kN\frac{V^2\rho}{\mu}. \tag{1.16}$$

To determine the unknown constant k, we postulate, a priori, that the rate of crack widening should be equal to the velocity of propagation of Rayleigh waves c; this postulate is based on the known results of the exact solutions of dynamic problems [1–3].

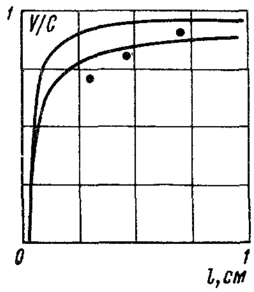


Fig. 1

Thus, from (1.16) we obtain the final expression

$$V = c\left(1 - \frac{K}{N}\right)^{1/2}. \tag{1.17}$$

In the case for which crack propagation takes place under the influence of a constant tensile stress, formula (1.17) can be written in the form

$$V = c\left(1 - \sqrt{\frac{l_0}{l}}\right)^{1/2}, \tag{1.18}$$

where l_0 is the length of an equilibrium crack and l is the crack length at a given instant.

Relation (1.18) is reproduced graphically in Fig. 1 (curve 1); formula (1.3) curve 2 is also plotted with three experimental points obtained in [6].

The problem of crack propagation (from rest) at a constant velocity was studied in [3, 9]. In this case, $l = l_0 + Vt$, or, if the initial crack length is neglected, $l = Vt$ so that the stress intensity coefficient

$$N = \frac{p\sqrt{Vt}}{\sqrt{2}}. \tag{1.19}$$

It follows, from (1.17), that the cohesion modulus should be proportional to $t^{1/2}$. This case, it was shown in [9], is realized if the crack tip region increases at a velocity which is constant for a given material. In these circumstances, instead of the cohesion constriction modulus K, we introduce a new material characteristic R which is related to K by

$$K = R\sqrt{2t}. \tag{1.20}$$

Substituting (1.20) and (1.19) into (1.17), we obtain

$$\frac{p\sqrt{c}}{R} = \frac{2}{\left(\frac{V}{c}\right)^{1/2}\left(1 - \frac{V^2}{c^2}\right)}. \tag{1.21}$$

Relation (1.21) is represented by a heavy solid line in Fig. 2, which also contains graphs obtained in [9].

As shown by the data in Figs. 1 and 2, formula (1.17)–which is based on simplifying assumptions–describes quite well the results obtained by other authors.

2. Equilibrium of a system of parallel cracks. This paragraph is devoted to the problem of equilibrium of an infinitely large number of parallel cracks (whose length are $2l$ and $2m$ and which are spaced at a distance h) in an elastic body to which a constant tensile stress p is applied at infinity (Fig. 3).

The walls of the cracks, which are regarded as slits, are free from stress. Let us solve this problem by an approximate method

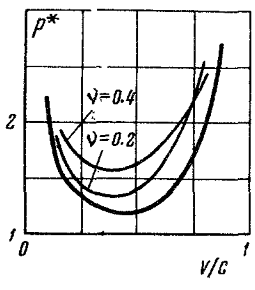


Fig. 2

postulated in [10]. By superposing omnidirectional uniform compression,

$$\sigma_y = \sigma_x = -p,$$

onto the existing stressed state i.e., we reduce the problem under consideration to a problem with zero stress at infinity and given stresses at the crack edges. Since the system is symmetrical relative to any straight line running along one of the cracks, the analysis can be limited to only one band $0 \le y \le h$ (Fig. 4). For the boundaries of this zone we have the following equations:

$$\sigma_x = \sigma_y = -p, \; y = 0, \; |x| \leqslant l; \; y = h, \; |x| \leqslant m \tag{2.1}$$

and, by virtue of the symmetry

$$\sigma_{xy} = 0, \; \frac{\partial v}{\partial x} = 0, \quad y = 0, \quad |x| > l; \quad y = h, \quad |x| > m. \tag{2.2}$$

Here and henceforth, the following generally accepted notation is used: $\sigma_x, \sigma_y, \sigma_{xy}$ denote stress tensor components, while u and v are displacement vector components.

Applying to the analysis of this stressed state the Kolosov-Muskhelishvili method [8], we obtain expressions relating stress and displacement to two analytical functions $\varphi(z)$ and $\psi(z)$:

$$\sigma_x + \sigma_y = 4\,\mathrm{Re}\,\varphi'(z),$$

$$\sigma_y - \sigma_x + 2i\sigma_{xy} = 2\,[z\varphi''(z) + \psi'(z)],$$

$$2\mu\,(u + iv) = \varkappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)}, \tag{2.3}$$

where μ is the Lamè constant and ν the Poisson ratio.

It is easy to show that conditions (2.1) and (2.2) are reduced, as a result of (2.3), to boundary conditions for one analytical function

$$\mathrm{Re}\,\varphi'(z) = -\frac{p}{2}, \quad y = 0, \quad |x| \leqslant l;$$

$$y = h, \quad |x| \leqslant m; \tag{2.4}$$

$$\mathrm{Im}\,\varphi'(z) = 0, \; y = 0, \; |x| > l; \; y = h, \; |x| > m. \tag{2.5}$$

This boundary problem is solved in the following way. A function

$$\zeta = l^{\pi z/h} \tag{2.6}$$

maps the band in question on the upper half-plane $\eta > 0$ with the correspondence of points illustrated in Fig. 5. For the function $f(\zeta) = \varphi'[z(\zeta)]$, we have the following boundary conditions:

$$\mathrm{Re}\,f(\zeta) = -\frac{p}{2}, \quad -a_1 < \xi < -\frac{1}{a_1},$$

$$a_2 < \xi < \frac{1}{a_2}, \quad \eta = 0, \tag{2.7}$$

$$\mathrm{Im}\,f(\zeta) = 0, \quad -\frac{1}{a_1} < \xi < a_2, \quad \xi > \frac{1}{a_2},$$

$$\xi < -a_1, \quad \eta = 0,$$

$$\left(a_1 = e^{\frac{\pi m}{h}}, \quad a_2 = e^{-\frac{\pi l}{h}}\right). \tag{2.8}$$

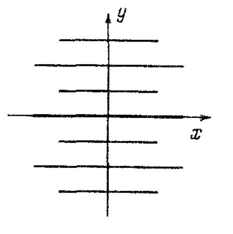


Fig. 3

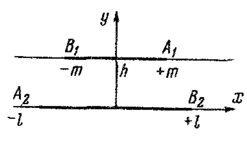


Fig. 4

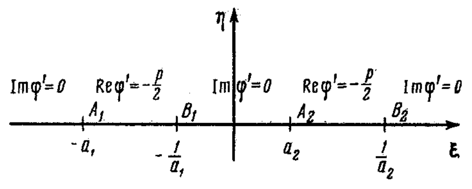


Fig. 5

A solution of this problem can be obtained with the aid of the Keldysh-Sedov formula [11].

In the class of functions which have integrable singularities at the ends of the segments A_1B_1 and A_2B_2 (Fig. 5), this solution is obtained, accurate to two arbitrary constants γ_0 and γ_1, and has the following form:

$$f(\zeta)=\frac{p}{2}\left[\frac{1}{g(\zeta)}-1\right]+$$

$$+\frac{\gamma_0+\gamma_1\zeta}{\sqrt{(\zeta+a_1)(\zeta+a_1^{-1})(\zeta-\alpha_2)(\zeta-a_2^{-1})}}$$

$$\left(g(\zeta)=\sqrt{\frac{(\zeta+a_1^{-1})(\zeta-a_2^{-1})}{(\zeta+a_1)(\zeta-a_2)}}\right). \tag{2.9}$$

The radical branches are chosen in such a way that they have positive value at $\zeta>1/a_2$. The value $f(\zeta)$ at infinity is equal to zero. Let us analyze the singularities at points A_1, B_1, A_2, and B_2. It is easily deduced from (2.9) and (2.6) that, when these points are approached from outside segments A_1B_1 and A_2B_2, the virtual part of the function approaches infinity, i.e., increases as (const $/\sqrt{s}$), where s is the distance from a given point (small in comparison with the crack length), namely:

at point A_1

$$\sigma_y=2\mathrm{Re}\,\varphi'=\frac{\gamma_0-\gamma_1 a_1}{\sqrt{a_1(a_1-a_1^{-1})(a_1+a_2)(a_1+a_2^{-1})}}\sqrt{\frac{h}{\pi s}}\equiv$$

$$\equiv\frac{N_1}{\sqrt{s}}, \tag{2.10}$$

at point B_1

$$2\mathrm{Re}\,\varphi'=p\sqrt{\frac{(a_1-a_1^{-1})(a_1^{-1}+a_2^{-1})}{a_1^{-1}(a_1^{-1}+a_2^{-1})}}\sqrt{\frac{h}{\pi s}}-$$

$$-\frac{\gamma_0-\dfrac{\gamma_1}{a_1}}{\sqrt{a_1^{-1}(a_1-a_1^{-1})(a_1^{-1}+a_2)(a_1^{-1}+a_2^{-1})}}\sqrt{\frac{h}{\pi s}}\equiv$$

$$\equiv\frac{N_1'}{\sqrt{s}}, \tag{2.11}$$

at point A_2

$$2\mathrm{Re}\,\varphi'=-\frac{\gamma_0+\gamma_1 a_2}{\sqrt{(a_1+a_2)(a_1^{-1}+a_2)(a_2^{-1}-a_2)a_2}}\sqrt{\frac{h}{\pi s}}\equiv$$

$$\equiv\frac{N_2}{\sqrt{s}}, \tag{2.12}$$

and at point B_2

$$2\mathrm{Re}\,\varphi'=p\sqrt{\frac{(a_2^{-1}+a_1)(a_2^{-1}-a_2)}{(a_2^{-1}-a_2)a_2^{-1}}}\sqrt{\frac{h}{\pi s}}+$$

$$+\frac{\gamma_0+\gamma_1/a_2}{\sqrt{(a_1^{-1}+a_1)(a_2^{-1}+a_1^{-1})(a_2^{-1}-a_2)a_2^{-1}}}\sqrt{\frac{h}{\pi s}}\equiv\frac{N_2'}{\sqrt{s}}. \tag{2.13}$$

To determine constants γ_0 and γ_1, we stipulate that the following equations (following from the physically evident condition of symmetry relative to the y axis are satisfied:

$$N_1=N_1',\ N_2=N_2'. \tag{2.14}$$

From (2.14) and (2.10)–(2.13) we obtain

$$\gamma_0=p(a_1,a_2-1),\ \gamma_1=p(a_2-a_1), \tag{2.15}$$

$$N_1=p\sqrt{\frac{h}{\pi}}\sqrt{\frac{a_2(a_1^2-1)}{(a_1+a_2)(1+a_1a_2)}},$$

$$N_2=p\sqrt{\frac{h}{\pi}}\sqrt{\frac{a_1(1-a_2^2)}{(a_1+a_2)(1+a_1a_2)}}.$$

In accordance with the theory of equilibrium cracks [7], the latter are at rest if the stress intensity coefficient is equal to the cohesion modulus K:

$$N_1=N_2=K. \tag{2.17}$$

Substituting (2.16) here, we find that

$$a_1=\frac{1}{a_2},\quad l=m,\quad N_1=N_2=p\sqrt{\frac{h}{2\pi}\,\mathrm{th}\,\frac{\pi l}{h}}. \tag{2.18}$$

Thus, the system of cracks under consideration can be at equilibrium if they are of the same length; at the same time,

$$\frac{p^2h}{2\pi}\,\mathrm{th}\,\frac{\pi l}{h}=K^2 \tag{2.19}$$

and, for $h/l\to\infty$, obtain an expression

$$p^2l=2K^2, \tag{2.20}$$

which coincides with the corresponding formula for a single crack.

In the second limiting case of closely situated cracks (i.e., for $h/l\to 0$) the equilibrium spacing between cracks does not depend on their length:

$$h=\frac{2\pi K^2}{p^2}. \tag{2.21}$$

Let us now assume that the system of cracks in Fig. 3 is present in a body and that tensile stresses such that $N_1>K$ and $N_2>K$ are instantaneously applied to this body.

The cracks will then start to propagate; the longer cracks will propagate faster, since $N_2>N_1$. After a while, the ratio of the crack length may become such thta, for the shorter cracks, N_1 will become smaller than K. In these circumstances, some of the shorter cracks will close up. In this case, the crack propagation rate should obviously be given by

$$V=c\left(\frac{K}{N}-1\right)^{1/2}. \tag{2.22}$$

3. Propagation of a system of cracks. Let us analyze the problem of the propagation of a system of cracks of two different lengths (shown in Fig. 3) from the standpoint of the approximate theory developed in paragraph 1. We have

$$\frac{dm}{dt}=c\left(1-\frac{K}{N_1}\right)^{1/2},\qquad \frac{dl}{dt}=c\left(1-\frac{K}{N_2}\right)^{1/2}, \tag{3.1}$$

where N_1 and N_2 are defined by formulas (2.16) which, after transformation, can be written in the form

$$N_1=p\left(\frac{h}{2\pi}\right)^{1/2}\left(\mathrm{sh}\,\frac{\pi m}{h}\,\mathrm{sch}\,\frac{\pi(m+l)}{2h}\,\mathrm{sch}\,\frac{\pi(m-l)}{2h}\right)^{1/2},$$

$$N_2=p\left(\frac{h}{2\pi}\right)^{1/2}\left(\mathrm{sh}\,\frac{\pi l}{h}\,\mathrm{sch}\,\frac{\pi(m+l)}{2h}\,\mathrm{sch}\,\frac{\pi(m-l)}{2h}\right)^{1/2}. \tag{3.2}$$

The initial crack length values are denoted by m_0 and l_0. The corresponding values of a_1, a_2, N_1, and N_2 are also denoted by a zero index. From (3.2), it follows that

$$\frac{N_2}{N_1}=\left(\mathrm{sh}\,\frac{\pi l}{h}\mathrm{csch}\,\frac{\pi m}{h}\right)^{1/2}, \tag{3.3}$$

so that, for $l>m$ and $N_2>N_1$.

If $N_{10}=K$ and $N_{20}=K$, the cracks do not widen. For $N_{10}<K<N_{20}$, the short cracks close up and the long cracks become wider.

Finally, for $N_{10} > K$ and $N_{20} > K$, widening of each type of crack takes place at rates given by Eqs. (3.1). Since these rates are not the same, in time, N_1 may become smaller than K and the widening of short cracks will cease.

Consider the case of cracks situated close to each other

$$\frac{l}{n} \gg \frac{m}{h} \gg 1. \qquad (3.4)$$

In this case, Eq. (3.1) can be written approximately in the form

$$\frac{dm}{dt} = c\left[1 - \frac{K}{p}\sqrt{\frac{\pi}{h}}\, e^{\frac{\pi(l-m)}{h}}\right]^{1/2}, \qquad (3.5)$$

$$\frac{dl}{dt} = c\left[1 - \frac{K}{p}\sqrt{\frac{\pi}{h}}\right]^{1/2}. \qquad (3.6)$$

Thus, the velocity of long cracks is constant while that of short cracks diminishes becoming zero when

$$l - m = \frac{2h}{\pi}\ln\frac{p}{K}\sqrt{\frac{\pi}{h}}. \qquad (3.7)$$

If all the cracks are of the same length, their velocity is given by a single equation,

$$\frac{dl}{dt} = c\left(1 - \frac{K}{N}\right)^{1/2}, \quad N = p\sqrt{\frac{h}{2\pi}}\sqrt{\operatorname{th}\frac{\pi l}{h}}. \qquad (3.8)$$

Let us analyze this equation in terms of stability. Assume that

$$l = l_0 + \varepsilon, \quad \varepsilon \ll l_0, \quad N_0 = p\sqrt{\frac{h}{2\pi}}\sqrt{\operatorname{th}\frac{\pi l_0}{h}}. \qquad (3.9)$$

Substituting (3.9) into (3.8) and performing the appropriate expansions with respect to small ε to terms of first order, we obtain (assuming thad $dl_0/dt \neq 0$)

$$\frac{d\varepsilon}{dt} = \frac{dl_0}{dt}\frac{K}{N_0 - K}\frac{\pi\varepsilon}{2h}\frac{1}{\operatorname{sh}^2\frac{\pi l_0}{h}}, \qquad (3.10)$$

which, taking into account (3.8), can be integrated in the form

$$\varepsilon = \frac{\varepsilon_0}{c}\frac{dl_0}{dt}, \qquad (3.11)$$

where ε_0 is the magnitude of the initial perturbation.

Thus, if the velocity of the system of identical cracks is, at some instant, increasing, a small disturbance will also increase; the motion, in this case, is unstable. If, however, the velocity is decreasing, the motion is stable. The propagation of a system of identical cracks has an instability of another kind. Let us imagine that each alternate crack increased its length by the same amount so that a system shown in Fig. 3 is obtained. The propagation of such a system is described by Eq. (3.1).

It is easy to show that the system of equations (3.1) is unstable. If small perturbances m and l are denoted by x and y,

$$m = l_0 - y, \qquad l = l_0 + x, \quad x, \quad y \ll l_0; \qquad (3.12)$$

expanding the right-hand portions of (3.1) into Taylor's series to terms of first order, we obtain the following system of linear differential equations:

$$\frac{dx}{dt} = A(\alpha y + \beta x), \quad \frac{dy}{dt} = -A(\alpha x + \beta y), \qquad (3.13)$$

where

$$A = \frac{\pi c}{4h}\frac{K}{N_0}\left(1 - \frac{K}{N_0}\right)^{-1/2}, \quad \alpha = \frac{1}{2}\operatorname{th}\frac{\pi l_0}{h},$$

$$\beta = \operatorname{cth}\frac{\pi l_0}{h} - \frac{1}{2}\operatorname{th}\frac{\pi l_0}{h}.$$

Its characteristic equation is in the form

$$s^2 - (\beta^2 - \alpha^2) = 0, \qquad (3.14)$$

since

$$\beta^2 - \alpha^2 = \operatorname{csch}^2\frac{\pi l_0}{h}.$$

Hence we arrive at the instability of Eq. (3.1).

4. **Brittle fracture under explosive loads.** Many materials that are not brittle in the usual sense of this term, fail by the mechanism of crack formation under the influence of suddenly applied loads exceeding their strength. Materials of this kind include metals. Figure 6 shows radiographs of an aluminum ring taken at various stages of its fracture (numbers ascribed to these radiographs indicate time in microseconds).

The matehematic theory of equilibrium cracks is not concerned with the problems associated with crack formation; it simply makes it possible to calculate the strength of a material in the presence of cracks of a given configuration. This leads to certain difficulties in interpreting the results obtained. In fact, in the above-considered problem of uniform straining (in tension) of an elastic-brittle body, the following four variants of crack formation (under the influence of identical loads applied at infinity) are possible.

1) A single equilibrium crack, of length given by Eq. (2.20), is formed.

2) A system of equilibrium cracks is formed; the length of the cracks and their spacing are given by Eq. (2.19).

3) A single growing crack is formed; its length is larger than the equilibrium value and the speed of its propagation is given by Eq. (1.21).

4) A system of cracks propagating at a velocity described by Eqs. (1.20) and (2.18) is formed.

To resolve these difficulties, let us utilize experimental data. The length of a single equilibrium crack is related to the strength of a given material. If the real strength of a material is σ^*, in accordance with (1.10) and (2.20) we have

$$\sigma_* \sim \sqrt{\frac{\gamma E}{l_0}}. \qquad (4.1)$$

It is known that, in terms of the order of magnitude, the theoretical strength σ_T is given by

$$\sigma_T \sim 0.1E. \qquad (4.2)$$

Following [12], we take

$$\gamma \sim 0.1\ b\ E, \qquad (4.3)$$

where b is the interatomic spacing.

Then, if α denotes a number showing how many times less the real strength of a material is than its theoretical strength, it follows from (4.1)-(4.3) that

$$l_0 \sim 10\alpha^2 b. \qquad (4.4)$$

For instance, in the case of mild steel and Duralumin $\alpha \sim 10^2$, so that

$$l_0 \sim 10^5 b = 10^{-3}\ \text{см}. \qquad (4.5)$$

Sometimes the term "dynamic strength" (which is higher than the strength determined in static tests) is used; in this case, one has in mind the fracture of a specimen at only one place, i.e., the formation and propagation of a single crack. In terms of the theory of equilibrium cracks, this means that the length of cracks corresponding to dynamic strength is smaller than the length in the case of static strength.

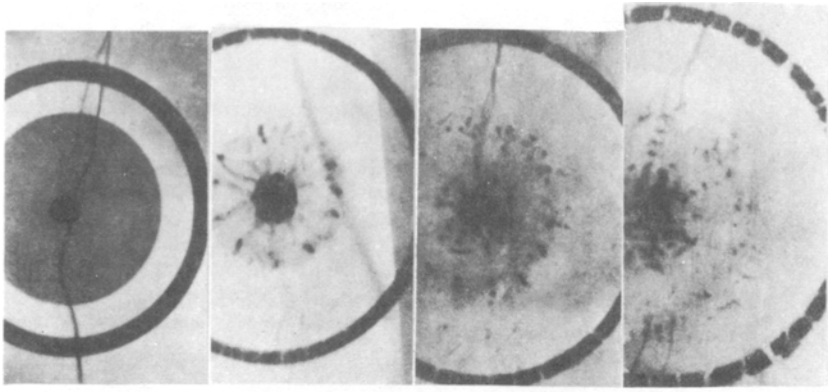
It should be noted that we have been discussing the tensile stress. If a sufficiently thin elastic ring is subjected to an internal pressure p_0, each element of the ring is under a tensile stress which is approximately given by

$$\sigma_\theta = p_0\frac{r}{\delta}. \qquad (4.6)$$

As shown by experiment (Fig. 6), if p_0 is sufficiently large so that $\sigma_\theta > \sigma_*$, numerous cracks are simultanenously formed in such a ring.

This means that the first and seocnd variants are not realized in this case and that the third and fourth variants are possible. As a result of the above-mentioned equilibrium instability, the regime can easily change from the second to the fourth variant.

When the ratio δ/r is small, each element of the ring may be regarded as part of a band of width δ under the influence of the tensile stress p applied at infinity. Going further and neglecting (in first approximation) the influence of the free boundaries, we obtain the problem of the propagation of a system of cracks in an infinite plane.



0 14 25 35

Fig. 6

The simplest model of such a system is the system of parallel cracks considered in paragraphs 2 and 3. If one accepts an additional hypothesis that the length of each crack is determined only by the material strength in accordance with formula (4.4), the distance between the initially formed microcracks is given by

$$\frac{p^2 h}{2\pi} \operatorname{th} \frac{\pi l_0}{h} = K^2.$$

As pointed out in paragraph 3, this system of cracks is unstable. Specifically, if each alternate crack is lengthened by the same amount, the longer cracks will grow faster and the shorter cracks will grow slower than previously. After a while, the shorter cracks will stop growing and will start closing up. The presence of short cracks may therefore be neglected. This leads to the formation of a new system of cracks with doubled spacing between them. Let us denote by N_2^* the stress intensity coefficient for a system of cracks with the spacing 2h

$$N_2^* = p\sqrt{\frac{h}{\pi}}\left(\operatorname{th}\frac{\pi l}{2h}\right)^{1/2}. \qquad (4.7)$$

Let us now equate N_2^* to N_2, defined by the second formula of (2.16). After transformation, the ration N_2^*/N_2 can be written in the form

$$\frac{N_2^*}{N_2} = \left(1 + \operatorname{sh}^2 \frac{\pi m}{2h} \operatorname{sch}^2 \frac{\pi l}{2h}\right)^{1/2}. \qquad (4.8)$$

If the crack length is e times larger than the initial length, this ratio differs from unity by 1–10%, depending on the ratio l_0/h. Thus, if the crack length at a given moment is l, the number of duplication acts n is equal to $\ln (l/l_0)$ and the distance between the cracks is $h = h_0 2^n$. Hence, we obtain

$$h = h_0 \left(\frac{l}{l_0}\right)^{\ln 2}. \qquad (4.9)$$

As an example, let us consider explosive fracture of a steel ring (Fig. 7). Fracture under relatively low loads is quite symmetrical and the scatter of the data for the dimensions of the resulting fragments is small.

Data on the length (H, mm) of fragments obtained as a result of the fracture of mild steel ring (diameter α = 80 mm, width ∂ = 9 mm, thickness δ = 10 mm) under the influences of a shock wave produced by an 80 g explosive charge are reproduced below:

No.	1	2	3	4	5	6	7	8	9	10	11	12
H, mm	49	37.5	32	33.5	41	34	31	18	10.5	15.5	21	19.5

Another set of data for rings 2.6, 4.6, and 9.0 mm thick is given below; these data include the experimental values of the length of fragments H_E (averaged for three experiments), theoretical values of the lengths H_T calculated from formula (4.9) and, for comparison, H_E and H_T in a dimensionless form.

δ, mm	H_E, mm	H_T, mm	$H_E/2.6$	$H_T/0.495$
2.6	10.2	4.95	1	1
4.6	17.7	7.4	1.7	1.5
9.0	28.5	12.8	2.7	2.6

The initial spacing between cracks for given p and σ_* is given by the following expression:

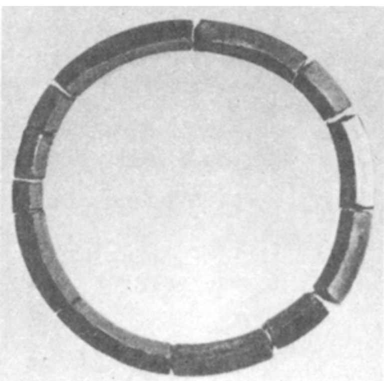
$$\frac{h}{\pi l_0} \operatorname{th} \frac{\pi l_0}{h} = \frac{\sigma_*^2}{p^2}.$$

If, according to this formula, the tensile stress p is 10% larger than the strength of a given material, the distance between initially formed cracks exceeds their length by one order of magnitude since, taking into account (4.5), we have

$$h_0 \sim 10^{-2} \text{ cm}. \qquad (4.11)$$

If, as mentioned above, the influence of free boundaries is neglected, the length H of a fragment is given by formula (4.9) in which the crack length l is taken to be equal to the band width (ring thickness) δ. For δ = 1 cm, the length of a fragment is also on the order of 1 cm which agrees with the experimental data (see above). As for the dependence of H on the ring width, it is satisfactorily described by formula (4.9) for h_0 = const. Generally speaking, if a ring is under quasi-static loads, it follows from (4.6) and (4.10) that h_0 should depend on the ring width. If, however, the initial stressed state is produced by a shock wave, the tensile stress and, consequently, h_0 do not depend on the ring width.

Thus, it has been shown that the above-analyzed model of crack formation is in quite good agreement with experiment. Naturally, such a model appears rather artificial; however, the process of "screening" small cracks by larger cracks has a clear physical meaning: the existence of a large and a small crack leads to stress relaxation in the vicinity of the latter.



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